Blow-up Rate Estimates for a Semilinear Heat Equation with a Gradient Term

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November 29, 2012

Abstract

We consider the the pointwise estimates and the blow-up rate estimates for the zero Dirchilet problem of the semilinear heat equation with a gradient term $u_t = \Delta u - |\nabla u|^2 + e^u$, which has been considered by J. Bebernes and D. Eberly in [1].

1 Introduction

Consider the following initial-boundary value problem

$$\left.\begin{array}{ll} u_t = \Delta u - h(|\nabla u|) + f(u), & (x,t) \in B_R \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & x \in B_R, \end{array}\right\} \tag{1.1}$$

where $f \in C^1(R)$, $h \in C^1([0,\infty))$, $f, h > 0$, $h' \geq 0$ in $(0,\infty)$, $f(0) \geq 0, h(0) = h'(0) = 0$,

$$|h(\xi)| \leq O(|\xi|^2), \tag{1.2}$$

$$sh'(s) - h(s) \leq Ks^q, \quad \text{for} \quad s > 0,\ 0 \leq K < \infty,\ q > 1, \tag{1.3}$$

$u_0 \geq 0$ is smooth, radial nonincreasing function, vanishing on ∂B_R, this means it satisfies the following conditions

$$\left.\begin{array}{ll} u(x) = u_0(|x|), & x \in B_R, \\ u_0(x) = 0, & x \in \partial B_R, \\ u_{0r}(|x|) \leq 0, & x \in B_R. \end{array}\right\} \tag{1.4}$$

Moreover, we assume that

$$\Delta u_0 + f(u_0) - h(|\nabla u_0|) \geq 0, \quad x \in B_R. \tag{1.5}$$

The special case

$$u_t = \Delta u - |\nabla u|^q + u|u|^{p-1}, \ p, q > 1 \tag{1.6}$$

was introduced in [2] and it was studied and discussed later by many authors see for instance [5, 12]. The main issue in those works was to determine for which p and q blow-up in finite time (in the L^∞-norm) may occur. It is well known that it occurs if and only if $p > q$ (see [5]). Equation (1.6) in R^n was considered from similar point of view, in this case blow-up in finite time is also known to occur when $p > q$, but unbounded global solutions always exist (see [12]). For bounded domains, it has been shown in [4] for equation (1.6) with general convex domain Ω that, the blow-up set is compact. Moreover if $\Omega = B_R$, then $x = 0$ is the only possible blow-up point and the upper pointwise rate estimate takes the following form

$$u \leq c|x|^{-\alpha}, \quad (x,t) \in B_R \setminus \{0\} \times [0,T),$$

for any $\alpha > 2/(p-1)$ if $q \in (1, 2p/(p+1))$, and for $\alpha > q/(p-q)$ if $q \in [2p/(p+1), p)$. We observe that $q/(p-q) > 2/(p-1)$ for $q > 2p/(p+1)$, therefore, the blow-up profile of solutions of equation (1.6) is similar to that of $u_t = \Delta u + u^p$ as long as $q < 2p/(p+1)$ (see [8]), whereas for q grater that this critical value, the gradient term induces an imprtant effect on the profile, which becomes more singular.

On the other hand, it was proved in [3, 4, 6, 13] that the upper (lower) blow-up rate estimate in terms of the blow-up time T in the case $q < 2p/(p+1)$ and $u \geq 0$, takes the following form

$$c(T-t)^{-1/(p-1)} \leq u(x,t) \leq C(T-t)^{-1/(p-1)}.$$

J. Bebernes and D. Eberly have considered in [1] a second special case of (1.1), where $f(s) = e^s, h(\xi) = \xi^2$, namely

$$\left.\begin{array}{ll} u_t = \Delta u - |\nabla u|^2 + e^u, & (x,t) \in B_R \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial B_R \times (0,T), \\ u(x,0) = u_0(x), & x \in B_R. \end{array}\right\} \tag{1.7}$$

The semilinear equation in (1.7) can be viewed as the limiting case of the critical splitting as $p \to \infty$ in the equation (1.6). It has been proved that, the solution of the above problem with u_0 satisfies (1.4) may blow up in finite time and the only possible blow-up point is $x = 0$. Moreover, if we consider the problem in any general bounded domain Ω such that $\partial\Omega$ is analytic, then the bow up set is a compact set. On the other hand, they proved that, if x_0 is a blow-up point for problem (1.7) with the finite blow-up time T; then

$$\lim_{t \to T^-} [u(x_0,t) + m \log(T-t)] = k,$$

for some $m \in Z^+$ and for some $k \in R$. The analysis therein is based on the observation that the transformation $v = 1 - e^{-u}$ changes the first equation in problem (1.7) into the linear equation $v_t = \Delta v + 1$, moreover, x_0 is a blow-up point for (1.7) with blow-up time T if and only if $v(x_0, T) = 1$.

In this paper we consider problem (1.7) with (1.4), our aim is to derive the upper pointwise estimate for the classical solutions of this problem and to find a formula for the upper (lower) blow-up rate estimate.

2 Preliminaries

The local existence and uniqueness of classical solutions to problem (1.1), (1.4) is well known by [7, 9]. Moreover, the gradient function ∇u is bounded as long as the solution u is bounded due to (1.2) (see [11]).

The following lemma shows some properties of the classical solutions of problem (1.1) with (1.4). We may denote for simplicity $u(r,t) = u(x,t)$.

Lemma 2.1. *Let u be a classical solution to the problem classical solution of problem (1.1) with (1.4). Then*

(i) *$u > 0$ and it is radial nonincreasing in $B_R \times (0,T)$. Moreover if $u_0 \not\equiv 0$, then $u_r < 0$ in $(0,R] \times (0,T)$.*

(ii) *$u_t \geq 0$ in $\overline{B}_R \times [0,T)$.*

Depending on Lemma 2.1, the problem (1.1) with (1.4) can be rewritten as follows

$$\left.\begin{array}{ll} u_t = u_{rr} + \frac{n-1}{r} u_r - h(-u_r) + f(u), & (r,t) \in (0,R) \times (0,T), \\ u_r(0,t) = 0, \quad u(R,t) = 0, & t \in [0,T), \\ u(r,0) = u_0(r), & r \in [0,R], \\ u_r(r,t) < 0, & (r,t) \in (0,R] \times (0,T). \end{array}\right\} \tag{2.1}$$

3 Pointwise Estimate

Inorder to derive a formula to the pointwise estimate for problem (2.1), we need first to recall the following theorem, which has been proved in [4].

Theorem 3.1. *Assume that, there exist two functions $F \in C^2([0,\infty))$ and $c_\varepsilon \in C^2([0,R]), \varepsilon > 0$, such that*

$$c_\varepsilon(0) = 0, c_\varepsilon^{'} \geq 0, \quad F > 0, F^{'}, F^{''} \geq 0, \quad in \quad (0,\infty), \tag{3.1}$$

$$f^{'}F - fF^{'} - 2c_\varepsilon^{'}F^{'}F + c_\varepsilon^2 F^{''}F^2 - 2^{q-1}Kc_\varepsilon^q F^q F^{'} + AF \geq 0, \quad u > 0, 0 < r < R, \tag{3.2}$$

where

$$A = \frac{c_\varepsilon^{''}}{c_\varepsilon} + \frac{n-1}{r}\frac{c_\varepsilon^{'}}{c_\varepsilon} - \frac{n-1}{r^2},$$

$\frac{c_\varepsilon(r)}{r} \to 0$ *uniformly on* $[0, R]$ *as* $\varepsilon \to 0$, *and*

$$G(s) = \int_s^\infty \frac{du}{F(u)} < \infty, \quad s > 0.$$

Let u *is a blow-up solution to problem (2.1), where* u_0 *satisfies*

$$u_{0r} \le -\delta, \quad r \in (0, R], \quad \delta > 0. \tag{3.3}$$

Suppose that, T *is the blow-up time. Then the point* $r = 0$ *is the only blow-up point, and there is* $\varepsilon_1 > 0$ *such that*

$$u(r,t) \le G^{-1}(\int_0^r c_{\varepsilon_1}(z)dz), \quad (r,t) \in (0, R] \times (0, T). \tag{3.4}$$

We are ready now to drive a formula to the pointwise estimate for the blow-up solutions of problem (1.7) with (1.4).

Theorem 3.2. *Let* u *be a blow-up solution to problem (1.7), assume that* u_0 *satisfies (1.4) and (3.3).Then the upper pointwise estimate takes the following form*

$$u(x,t) \le \frac{1}{2\alpha}[\log C - m\log(r)], \quad (r,t) \in (0, R] \times (0, T),$$

where $\alpha \in (0, 1/2], C > 0, m > 2$.

Proof. Let $c_\varepsilon = \varepsilon r^{1+\delta}$, where $\delta \in (0, \infty)$.

It is clear that c_ε satisfies the assumptions (3.1) in Theorem 3.1, so that (3.2) becomes

$$\begin{aligned} &f^{'}F - fF^{'} - 2\varepsilon(1+\delta)r^\delta F^{'}F + \varepsilon^2 r^{2+2\delta}F^{''}F^2 \\ &-2^{q-1}K\varepsilon^q r^{q+\delta q}F^qF^{'} + \frac{\delta(n+\delta)}{r^2}F \ge 0, \ u > 0, 0 < r < R. \end{aligned} \tag{3.5}$$

For the semilinear equation in (1.7) it is clear that $K \ge 1, q = 2$. To make use of Theorem 3.1 for problem (1.7), assume that

$$F(u) = e^{2\alpha u}, \quad \alpha \in (0, 1/2].$$

It is clear that F satisfies all the assumptions (3.1) in Theorem 3.1. With this choice of F the inequality (3.5) takes the form

$$\begin{aligned} &(1-2\alpha)e^{(1+2\alpha)u} + 4\alpha^2\varepsilon^2 r^{2(1+\delta)}e^{6\alpha u} + \frac{\delta(n+\delta)}{r^2}e^{2\alpha u} \ge \\ &4\alpha\varepsilon(1+\delta)r^\delta e^{4\alpha u} + 4\alpha\varepsilon^2 r^{2(1+\delta)}e^{6\alpha u}, \quad u \ge 0, 0 < r \le R \end{aligned}$$

provided $\alpha \leq \frac{1}{2+4\varepsilon R^{\delta}(1+\delta)}$.

Define the function G as in Theorem 3.1 as follows

$$G(s) = \int_s^{\infty} \frac{du}{e^{2\alpha u}} = \frac{1}{2\alpha e^{\alpha s}}, \quad s > 0.$$

Clearly,

$$G^{-1}(s) = -\frac{1}{2\alpha} \log(2\alpha s), \quad s > 0.$$

Thus (3.4) becomes

$$u(r,t) \leq \frac{1}{2\alpha}[\log C - m \log(r)], \quad (r,t) \in (0,R] \times (0,T),$$

where $C = \frac{2+\delta}{2\varepsilon\alpha}, \quad m = 2+\delta.$ □

Remark 3.3. Theorem 3.2 shows that, with choosing $\alpha = 1/2$, the upper pointwise estimate for problem (1.7) is the same as that for $u_t = \Delta u + e^u$, which has been considered in [8]. Therefore, the gradient term in problem (1.7) has no effect on the pointwise estimate.

4 Blow-up Rate Estimate

Since under the assumptions of Theorem 3.2, $r = 0$ is the only blow-up point for the problem (1.7), therefore, in order to estimate the blow-up solution it suffices to estimate only $u(0,t)$. The next theorem, which has been proved in [4], considers the upper blow-up rate estimate for the general problem (1.1).

Theorem 4.1. *Let u be a blow-up solution to problem (1.1), where $u_0 \in C^2(\overline{B}_R)$ and satisfies (1.4), (1.5). Assume that T is the blow-up time and $x = 0$ is the only possible blow-up point. If there exist a function, $F \in C^2([0,\infty))$ such that $F > 0$ and $F', F'' \geq 0$ in $(0,\infty)$, moreover,*

$$f'F - F'f + F''|\nabla u|^2 - F'[h'(|\nabla u|)|\nabla u| - h(|\nabla u|)] \geq 0, \ \textit{in } B_R \times (0,T), \quad (4.1)$$

then the upper blow rate estimate takes the from

$$u(0,t) \leq G^{-1}(\delta(T-t)), \quad t \in (\tau, T),$$

where $\delta, \tau > 0$, $G(s) = \int_s^{\infty} \frac{du}{F(u)}$.

For problem (1.7), if one could choose a suitable function F that satisfies the conditions, which have stated in Theorem 4.1, then the upper blow-up rate estimate for this problem would be held.

Theorem 4.2. *Let u be a blow-up solution to problem (1.7), where $u_0 \in C^2(\overline{B}_R)$ and satisfies (1.4), (3.3) and the monotonicity assumption*

$$\Delta u_0 + e^{u_0} - |\nabla u_0|^2 \geq 0, \quad x \in B_R,$$

suppose that T is the blow-up time.Then there exist $C > 0$ such that the upper blow-up rate estimate takes the following form

$$u(0,t) \leq \frac{1}{\alpha}[\log C - \log(T-t)], \quad 0 < t < T, \ \alpha \in (0,1].$$

Proof. Let

$$F(u) = e^{\alpha u}, \quad \alpha \in (0,1].$$

It is clear that the inequality (4.1) becomes

$$(1-\alpha)e^{(1+\alpha)u} + \alpha^2 e^{\alpha u}|\nabla u|^2 - \alpha e^{\alpha u}|\nabla u|^2 \geq 0,$$

which holds for any $\alpha \in (0,1]$.

Set

$$G(s) = \int_s^\infty \frac{du}{e^{\alpha u}} = \frac{1}{\alpha e^{\alpha s}}, \quad s > 0.$$

Clearly,

$$G^{-1}(s) = -\frac{1}{\alpha}\log(\alpha s), \quad s > 0.$$

From Theorem 4.1 there is $\delta > 0$ such that

$$u(0,t) \leq \frac{1}{\alpha}[\log(\frac{1}{\alpha\delta}) - \log(T-t)], \quad \tau < t < T.$$

Therefore, there exist a positive constant, C such that

$$u(0,t) \leq \frac{1}{\alpha}[\log C - \log(T-t)], \quad 0 < t < T.$$

□

Next, we consider the lower blow-up rate for problem (1.7), which is much easier than the upper bound.

Theorem 4.3. *Let u be a blow-up solution to problem (1.7), where u_0 satisfies (1.4) and (3.3). Suppose that T is the blow-up time.Then there exist $c > 0$ such that the lower blow-up rate estimate takes the following form*

$$\log c - \log(T-t) \leq u(0,t), \quad 0 < t < T.$$

Proof. Define

$$U(t) = u(0,t), \quad t \in [0,T).$$

Since u attains its maximum at $x = 0$,

$$\Delta U(t) \leq 0, \quad 0 \leq t < T.$$

From the semilinear equation in (1.7) and above, it follows that

$$U_t(t) \leq e^{U(t)} \leq \lambda e^{U(t)}, \quad 0 < t < T, \tag{4.2}$$

for $\lambda \geq 1$. Integrate (4.2) from t to T, we obtain

$$\frac{1}{\lambda(T-t)} \leq e^{u(0,t)}, \quad 0 < t < T.$$

It follows that

$$\log c - \log(T-t) \leq u(0,t), \quad 0 < t < T,$$

where $c = 1/\lambda$. □

Remark 4.4. Theorem 4.3 (Theorem 4.2, where $\alpha = 1$) show that, the lower (upper) blow-up rate estimate for problem (1.7) is the same as for $u_t = \Delta u + e^u$, which has been considered in [8], therefore, we conclude that, the gradient term in problem (1.7) has no effect on the blow-up rate estimate.

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